Effect of Advertisement to the Simple Recommendation Model

Uzay Cetin[1, 2] and Haluk O. Bingol[1]

[1]*Department of Computer Engineering, Bogazici University, Istanbul*
[2]*Department of Computer Engineering, Istanbul Gelisim University, Istanbul*

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A spread model, based on the Simple Recommendation Model, is developed, simulated and analyzed using Markov Chains. Our model incorporates two distinct ways of information spread, namely, transmission from one person to another through interpersonal communication via recommendations or from mass media via advertisements. In numerical simulations it is observed that advertisement is more effective than word-of-mouth when the memory is limited. This observation is analytically explained for memory size of 1. Our analytical predictions are in agreement with the numerical simulations.

I. INTRODUCTION

Word-of-mouth recommendations by friends make products socially contagious. Research on social contagion can provide answers to the question of how things become popular. Gladwell states, "Ideas, products, messages and behaviors spread like viruses do" [6]. He claims that the best way to understand the emergence of fashion trends is to think of them as epidemics. Infectious disease modeling is also very useful for understanding opinion formation dynamics. Various models have been proposed to examine this relationship [1, 2, 4, 7, 12, 13]. Specifically, the transmission of ideas within a population is treated as if it were the transmission of an infectious disease. There exists recent works whose essential assumption is the fact that an old idea is never repeated once abandoned [3, 9]. In other words, agents become immune to older ideas like in the susceptible-infected-recovered (SIR) model. However, behaviors, trends, etc, can occur many times over and over again in an individual's life.

A good discussion on how stochastic evolutionary dynamics applies to critical real-world problems, including diffusion of epidemics, can be found in Nowak's book [11]. He describes the Moran process as the simplest possible stochastic model to study selection in a finite population. In the Moran process, the total population size is strictly constant. At each iteration, two individuals are selected. One reproduces to increase its number in the population. The other one is eliminated.

Bingol's Simple Recommendation Model (SRM) investigates how to become popular among agents with limited memory size and analyzes the word-of-mouth effect in its simplest form [2].

In this article, SRM is modified in order to incorporate two distinct ways of information spread. In addition to word-of-mouth spreading, we also incorporate an advertisement mechanism. We explore the connection between the diffusion of epidemics and becoming popular, as well as analyze the impact of advertisement. Moreover, we also provide an analytical explanation that can easily predict the steady state of the system by specifying a corresponding Markov Chain. Unlike to [9], our analytical derivations use discrete time-steps rather than continuous time. Distinctively, our model can be considered as a variant of susceptible-infected-susceptible (SIS) model.

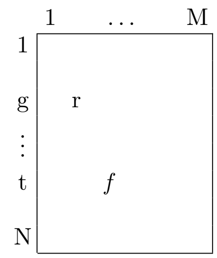


FIG. 1: The state of an agent is the content of its memory. Similarly, the state of the system is the memories of all the agents. Conceptually, system can be thought as a $N \times M$ matrix in which each row represents the memory content of a particular agent.

A. The Simple Recommendation Model

In the SRM, agents know items. Let $\mathcal{N} = \{1, 2, \ldots, N\}$ be the set of N agents and $\mathcal{I} = \{1, 2, \ldots, I\}$ be the set of I items.

Here item should be taken as political ideas, fashion trends, or cultural products as in the case of Herdagdelen [8], rather than unique objects such as a painting of Picasso. So there are enough of them for everybody to have, if they wanted to. Therefore there is a competition among items for market share.

Agent's memory size is limited in the sense that they cannot know the entire set of items but a small fraction of it, that is, $M \ll I$. When an agent is asked, she recommends an item selected at random from what she knows. The opponent agent accepts what she proposes without questioning. The memory content of an agent i, denoted by $m(i)$, consists of a set of items that agent i knows. Formally, we say agent i *knows* item j iff $j \in m(i)$. For the sake of simplicity, SRM has two abstractions: (i) In-

stead of agents recommending items, they recommend agents. Hence, $\mathcal{N} = \mathcal{I}$. (ii) All agents has the same memory size, that is $|m(i)| = M$ for all $i \in \mathcal{N}$. This study is considered to be based on a more general case of SRM in which agents recommending items. The sets $\mathcal{N}$ and $\mathcal{I}$ is distinguished to be the sets of agents and items, respectively but their size is kept equal, $N = I$.

Fame of an item is defined to be the ratio of agents who know the corresponding item in the population. In a way fame is the market share an item gets. In the SRM, agents learn items solely via recommendations.

The following convention is used: g, t, r, f represent the *giver* and *taker* agents, the *recommended* and *forgotten* items, respectively. Fig. 1 visualizes the system. Row i contains M items that agent i knows. The recommendation process takes place through a series of steps. (i) A pair of agents is selected as giver-taker agents at random. (ii) A random item is chosen for recommendation from the giver agent's memory. (iii) The recommendation process is accomplished by putting the recommended item into the taker's memory, that is, *t learns r*. (iv) In order to have a space for the recommended, a randomly selected item in the taker's memory is *forgotten*, that is, eliminated.

SRM differs from many previous models by its emphasis on the memory organization. It must be noted that if recommended item is already known by the taker agent, recommendation has no effect. Because it does not cause any change in the memory content of the taker agent. The state where every agent has exactly the same memory content is called an *absorbing state*. It is a final state where the system cannot leave. When system reaches to an absorbing state, the same M items are known by every agent and the remaining $N - M$ items are completely forgotten.

II. PROPOSED MODEL

The main interest of the SRM is to reveal the relation between the limited memory size of agents and the popularity of an item. In this model, initially, every item has an equal chance to be popularized. We extend the SRM to answer the following question: What happens if some items are promoted?

We made a slight change in the existing rules of SRM. We introduce a new parameter p that enables t to choose between what g recommends and what is promoted to the over-all population. Let $\mathcal{A} = \{N+1, N+2, \ldots, N+A\}$ be the set of A advertised items.

At each recommendation, the unique advertised item $a \in \mathcal{A}$ is promoted with a probability p. Note that for $A > 1$ cases, $a \in \mathcal{A}$ is selected at random. In this current work, the number of advertised items is set to one, namely, $A = 1$. Hence, a is unique. The SRM is so modified that the taker learns α where α is a with probability p and r with probability $1 - p$. Hence, the recommendation process is composed of the following steps:

1. g and t are selected
2. g recommends r
3. α is set to a with probability p, to r with probability $1 - p$
4. t learns α
5. t forgets f

Where all selections are at random.

Similar to the early work by Bass [1], our model incorporates both innovative and imitative behavior of agents. The advertisement probability p can be regarded as the explicit innovative adoption rate, whereas the recommendation process is the implicit imitative behavior of agents as a result of social pressure.

III. NUMERICAL SIMULATIONS

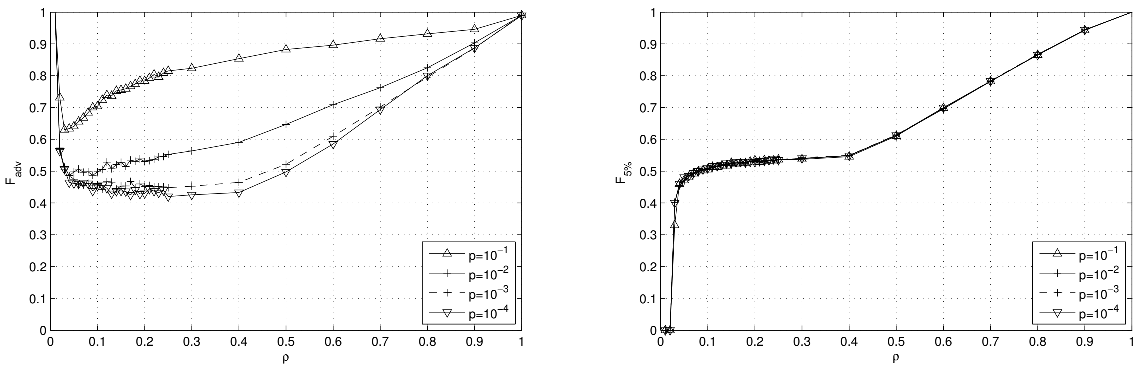
A. Settings

We investigate the fame of the advertised item in a population of $N = 100$ agents. Our underlying topology in this study is a complete graph. Individuals are equally likely to come into contact with any other. Each realization is started with the parameter p which is the fixed probability of the successful advertisement. At the initial configuration, the number of agents that knows the advertised item is zero. ν is the average iteration number for an agent. That is to say we have $\nu \times N$ as the total amount of iteration number. ν is fixed to 10^6. Clearly, if we simulate long enough, the system will fall into one of its absorbing states. By setting $\nu = 10^6$, we investigate near absorbing state behavior.

Usually, an individual does not know all the items rather she can be aware of a small fraction of them. Hence, we particularly concentrate on small memory size. We define ρ as the ratio of memory size to population size, that is, $\rho = \frac{M}{N}$. For $N = 100$, a $\rho = 0.01 - 0.25$ range with increments of 0.01 is studied. Additionally, a $\rho = 0.3 - 1$ range with increments of 0.1 is simulated in order to see the behavior of the system for larger memory sizes. We present the average results for different $p \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$ values over 100 realizations. In these simulations, effects of changing memory size to the item's fame are investigated.

B. Observations

How do people form their opinions? Do they imitate their friends by getting recommendations or do they follow the advertised items? The simulations are conducted to answer the question: Under which conditions advertisement mechanism outperforms the recommendation process? In order to make a comparison, suppose an ordering over items according to their fames, we define $F_{5\%}$ as the top 5 percent of the most famous items and F_{adv} as the fame of the advertised item. If $F_{adv} > F_{5\%}$,



(a)Fame of the advertised item, $N = 100$

(b)The top 5 percent of the most famous items, $N = 100$

FIG. 2: Fame of the advert vs. the top 5% of the population

we can make a quantitative conclusion that the fame of the advertised item is among the highest 5 percent of all items.

In Fig. 2 fame is plotted as memory size increases for various advertisement probabilities. The fame of the advertised item and the average fame of the top 5% items are plotted in Fig. 2(a) and Fig. 2(b), respectively. Results show that the advertisement mechanism reinforced with the recommendation process, plays very important role in the adoption of new ideas.

The most important model ingredient is the memory size of agents. We consider that there exists three types of memory size. We define, small memory size, if $\rho < 0.05$, moderate memory size if $0.05 < \rho < 0.35$ and large memory size if $\rho > 0.35$. When memory size is sufficiently large, no forgotten items are observed. As memory size decreases, completely forgotten item appears. Further decrease in M, increases the number of completely forgotten items. During this process, a few items become more popular in expense of the majority being forgotten. The findings from our simulations are also in accord with Bingol's early results which showed that the fame emerges as a result of the small memory size.

In addition to this, by comparing Fig. 2(a) and Fig. 2(b), we can determine the conditions under which the advertisement mechanism is more favorable than the recommendation process. In the general case, we observe that for greater p, the system always favors the popularity of the advertisement regarding to the top 5 percent of the most famous items of the population. Compare data for $p = 10^{-1}$ in Fig. 2(a) with the data sets in Fig. 2(b). On the other hand, for smaller p, we observe that the system can still significantly favor the popularity of the advertisement in the case of small memory size. For moderate and large memory size, as ρ increases, both F_{adv} and $F_{5\%}$ increases. Especially, F_{adv} and $F_{5\%}$ display a similar progress when advertisement probability is smaller. Furthermore, $F_{5\%}$ is insensitive to p for all kinds of memory sizes.

Note that, as the memory size of the agents gets smaller, it gets easier to outperform the recommendation process. When $\rho < 0.05$, we figure out that F_{adv} is much more greater than $F_{5\%}$. This also holds for $F_{10\%}$ and $F_{20\%}$ even for $F_{M\%}$ but due to space limitations, we are unable to give related results for them in more detail. This indicates the fact that the fame is stolen by advertisement for small memory size. Moreover, an unexpected behavior is observed when memory size is equal to one, that is, $\rho = 0.01$. In our initial configuration, there was no agent who knows the advertised item. However, at the end of our simulations, even when advertisement probabilities are vanishingly small, the only famous item is the one who is promoted by the advertisement mechanism. Hence this does not depend on the advertisement probability. The success of the advertisement can be entirely attributed to the shortage of the memory capacity of agents. We will deal with the analysis of this phenomenon in general, in the next section.

IV. ANALYTICAL DERIVATION OF THE ADVERTISED ITEMS POPULARITY

Opinions can spread from one person to another like diseases. The study of how ideas spread is often referred to as *social contagion* [5]. Here the advertised item can be considered as virus. The state of an agent can be susceptible or infected. The agents who adopted the advertisement's intimation are called *infected* and those who do not have the disease yet but can catch it, are called *susceptible*. Agent i is infected if $a \in m(i)$ and is susceptible if $a \notin m(i)$. Then $a \in m(g)$ represents the infected giver agent, who has adopted the advertiser's intimation and $a \notin m(g)$ represents the susceptible giver agent, who does not yet.

Let's define S_i as the state in which the number of the infected agents is i. Then, the population is composed of i infected agents and $N - i$ susceptible agents at state

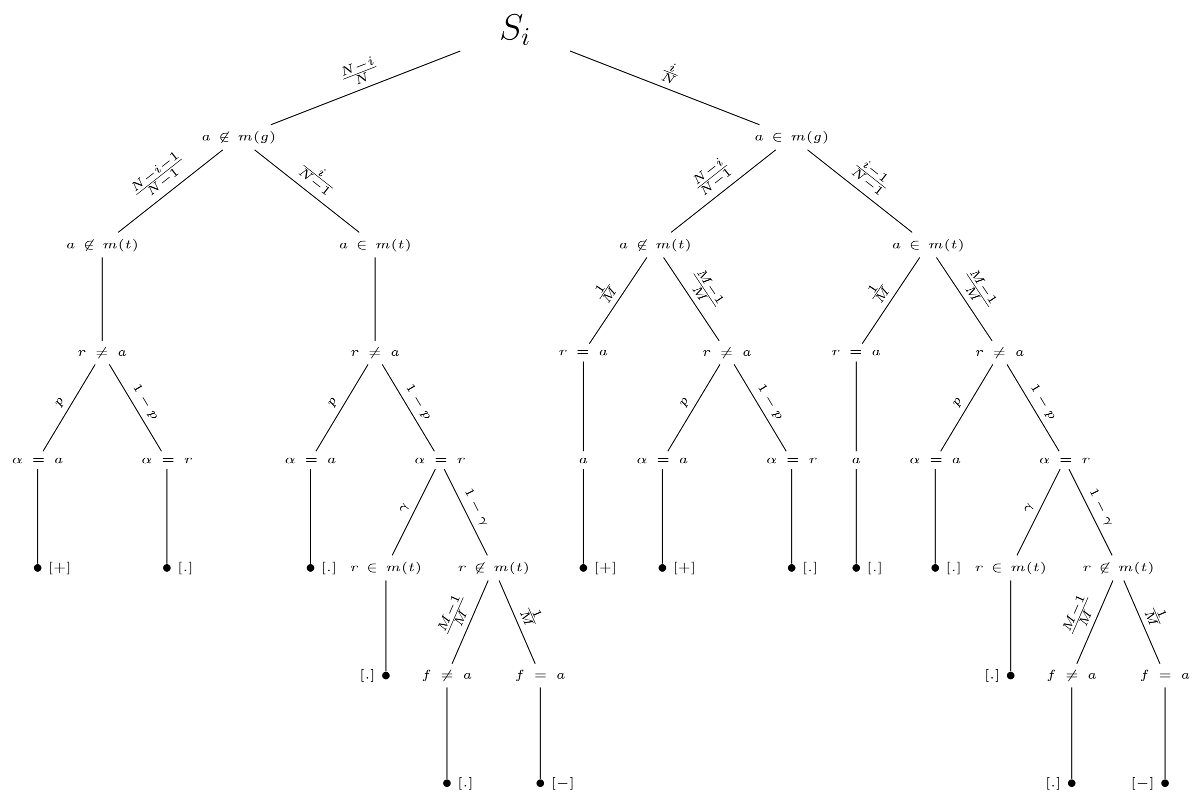


FIG. 3: Corresponding tree diagram for the undertaken simulation

S_i. Each iteration of the algorithm can be thought of as consisting of a series of random selections. There are N possible agents to select as g and $N-1$ to select as t. The items r and f are selected from M possible candidates and a is selected from A possible advertised items.

It is often found to be useful to use a tree diagram when studying probabilities of events relating to experiments that take place in stages. In this context, we suppose that we are in state S_i and we build the tree diagram shown in Fig. 3 as follows:

1. The giver can be in one of the two states $\{$*susceptible, infected*$\}$. So the first stage in the tree diagram has two branches. The probability of selecting an infected giver is $\mathbb{P}(a \in m(g)) = \frac{i}{N}$ and the probability of selecting a susceptible giver is $\mathbb{P}(a \notin m(g)) = \frac{N-i}{N}$.

2. Second stage, again, branches out according to the case whether we select a susceptible or an infected taker. This reflects a conditional probability, for example, the probability of selecting an infected taker, given that we have already selected an infected giver is $\mathbb{P}(a \in m(t) | a \in m(g)) = \frac{i-1}{N-1}$. In the same manner, the probability of each branch is written on the branch and the outcome is written at the end of the branch.

3. In the third stage, after a giver is matched with a taker, we look for what giver recommends. As we pointed out formerly, each agent's memory is composed of M items, so the infected giver attempts to spread the virus to the taker agent with a probability of $\mathbb{P}(r = a) = \frac{1}{M}$.

4. Then the fourth stage illustrates the taker's decision. Let's call it α. The taker agent either follows the advertisement with probability $\mathbb{P}(\alpha = a) = p$ or he accepts the recommended item with probability $\mathbb{P}(\alpha = r) = 1 - p$.

5. The most crucial step of this diagram is at the fifth stage. Here, we introduce a parameter γ, which denotes the probability of recommended item to be already known by the taker agent. In this case, with probability $\gamma = \mathbb{P}(r \in m(t))$, the taker agent does not learn anything new, so he does not need to forget anyone.

6. The remaining stages are about what to forget.

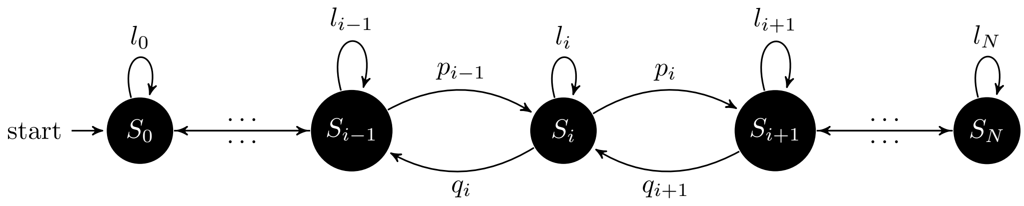


FIG. 4: A variant of random walk, we define S_i to be the state in which there exists i infected agents in the over-all population.

The leafs are marked with [$-$], [.] and [+]. If recommendation takes a path to a [$-$] leaf, that the system makes a $S_i \to S_{i-1}$ transition. Similarly, a [+] leaf corresponds to state transition $S_i \to S_{i+1}$. Leafs with [.] mark corresponds to state transitions of $S_i \to S_i$.

If a susceptible taker turns out to be infected at the end of iteration, then infectious is spread to one more agent. Namely, system moves from state S_i to S_{i+1}. This is shown by [+] leafs. On the other hand, if an infected taker turns out to be susceptible at the end iteration, then infectious is removed from one agent. So, in this case, system moves from state S_i to S_{i-1}. This is shown by [$-$] leafs. Otherwise, we stay at the current state, S_i. This is shown by [.] leafs.

Let's consider that we follow the branches on the right most path of the tree diagram. So, an infected giver ($a \in m(g)$) is matched with an infected taker ($a \in m(t)$) and what giver recommends is not the advert ($r \neq a$) and taker decides to get recommendation from him ($\alpha = r$). In this case, giver can select the advertised item to be forget forgotten ($f = a$) with probability $(1-\gamma)\frac{1}{M}$. This results in decrease of the number of the infected agents in the population. In other words, system moves from state S_i to S_{i-1}.

A. Connection to Random Walks

Random walks can be used as a model of diffusion. The state of the system can be represented as a number of agents that know the advertised item. Then the state space is composed of $N+1$ states,$\{S_0, S_1, \ldots, S_N\}$ where state S_i is the state that the advertised item is known by exactly i agents. Fig. 4 represents this random walk. Note that, S_0 and S_N are reflecting boundaries.

Let's use the following notation for representing the transition probabilities from state S_i.

$$\begin{aligned} q_i &= \mathbb{P}(S_i \to S_{i-1}) \\ l_i &= \mathbb{P}(S_i \to S_i) \\ p_i &= \mathbb{P}(S_i \to S_{i+1}) \end{aligned}$$

p_i is the transition probability from S_i to S_{i+1}, q_i is the transition probability from S_i to S_{i-1}, l_i is the probability of staying at the current state S_i. Since all probabilities of a tree diagram should add up to 1, we have:

$$q_i + l_i + p_i = 1$$

Transition probabilities can easily be calculated by multiplying the probabilities along the branches of our tree diagram.

$$\begin{aligned} p_i &= \frac{N-i}{N}\frac{N-i-1}{N-1}p \\ &+ \frac{i}{N}\frac{N-i}{N-1}\left(\frac{1}{M} + \frac{M-1}{M}p\right) \qquad (1) \\ q_i &= \frac{N-i}{N}\frac{i}{N-1}(1-p)(1-\gamma)\frac{1}{M} \\ &+ \frac{i}{N}\frac{i-1}{N-1}\frac{M-1}{M}(1-p)(1-\gamma)\frac{1}{M} \qquad (2) \\ l_i &= 1 - (p_i + q_i). \qquad (3) \end{aligned}$$

The random walk seen in Fig. 4 is used to estimate the underlying probability distribution over the states which correspond to the number of infected agents.

B. Extreme case, $M = 1$

The equations for the state transition probabilities simplify for the extreme case of $M = 1$. In Fig. 4, a path that contains $M-1$ term becomes ineffective and the corresponding probability becomes 0. Since there is only one memory location, γ calculation also becomes manageable. Therefore the probabilities can be obtained as follows.

Consider the transition of $S_i \to S_{i+1}$. Setting $M-1 = 0$ in Eq. 1, one obtains

$$p_i = \frac{N-i}{N}\frac{N-i-1}{N-1}p + \frac{i}{N}\frac{N-i}{N-1}. \qquad (4)$$

Now, consider the transition of $S_i \to S_{i-1}$. Setting $M = 1$ in the above equation for q_i, one obtains

$$q_i = \frac{N-i}{N}\frac{i}{N-1}(1-p)(1-\gamma). \qquad (5)$$

For this particular case, it is possible to determine the

value of γ. Note that in Fig. 4 there are two paths that terminate at a $[-]$ leaf. Due to $M-1=0$, $\mathbb{P}(r \neq a) = \frac{M-1}{M} = 0$. Hence the right path to a $[-]$ leaf disappears which is what we obtained in Eq. 5. There is only the left path remains, namely, the path composes of the nodes $S_i, a \notin m(g), a \in m(t), r \neq a, \alpha = r$. At this point we need the value of γ which can be obtained by the following observations. In this path g does not know a while t does. Since the memory has only one slot, g and t do not know any item common. Therefore, the recommended item by the giver cannot be known by the taker. Hence the probability of this is 0, that is, $\gamma = 0$. So the path takes the right branch to node $r \notin m(t)$. Using $M-1=0$ again, the path reaches to $[-]$ leaf. So the corresponding probability is

$$q_i \;=\; \frac{N-i}{N}\frac{i}{N-1}(1-p). \tag{6}$$

These values of p_i and q_i have an unexpected consequence. In our extreme case of $M=1$, we have

$$\frac{p_i}{q_i} = \frac{i+(N-i-1)p}{i(1-p)} = 1 + \frac{N-1}{i}\frac{p}{1-p} > 1.$$

That is, the system inevitable drifted to the right till it gets to the state S_N. Using Eq. 6, we have $q_N = 0$. Therefore, once the state S_N is reached, the systems stays there forever. In other words, S_N becomes an absorbing state for $M=1$. Note that $p_0 > 0$ for $p > 0$. So S_N is the only absorbing state of the system. As any stochastic system does, this system is bound to be stuck at S_N.

State S_N means all the population is infected. Notice that this is independent of the value of the probability of advertisement p. Even very small values of p are enough for entire population to get the advertised item, along as p is nonzero.

C. General case, $M > 1$

We generalize our problem to the case when $M > 1$. Note that we still consider that there exists only one advertised item, namely, $A = 1$. Markov chain seen in Fig. 4, can be considered as a special form of Markov process known as the birth-death processes [10]. This implies that there exists a unique stationary state probability distribution denoted by π. Our proposed algorithm to find out π consists of two stages: (i) Represent the problem in the form of a Markov Matrix, Θ and (ii) Figure out the steady state using the spectral properties of Θ.

In order to construct a Markov matrix, the state of the system is characterized by one variable, that is, the number of infected agents, denoted by i. As a result, S_i again denotes the state in which the number of the infected agents is i. We investigate the probabilities of passing from one state to another. Let q_i, p_i and l_i be the transition probabilities of $S_i \to S_{i-1}$, $S_i \to S_{i+1}$ and $S_i \to S_i$, respectively. All other remaining transitions probabilities are equal to zero. This corresponds to an $(N+1)\times(N+1)$ tridiagonal matrix whose columns add up to 1. As an example, for $N=5$,

$$\Theta = \begin{bmatrix} l_0 & q_1 & 0 & 0 & 0 & 0 \\ p_0 & l_1 & q_2 & 0 & 0 & 0 \\ 0 & p_1 & l_2 & q_3 & 0 & 0 \\ 0 & 0 & p_2 & l_3 & q_4 & 0 \\ 0 & 0 & 0 & p_3 & l_4 & q_5 \\ 0 & 0 & 0 & 0 & p_4 & l_5 \end{bmatrix}$$

Θ is a stochastic matrix and we know from linear algebra that Θ and $\Theta^\top$ have the same eigenvalues. It is easy to see that $v = [1 \; \ldots \; 1]^\top$ is the eigenvector of $\Theta^\top$ with eigenvalue $\lambda = 1$. Hence, $\lambda = 1$ is also eigenvalue of Θ but with a different eigenvector π, that is,

$$\pi = \Theta\pi$$

where $\pi = [\pi_0 \; \pi_1 \; \ldots \; \pi_N]^\top$, is a $(N+1)\times 1$ column vector. We can interpret π as a stationary distribution of the walk, for which π_i is the probability of being at any particular state S_i. It can be easily shown that the stationary distribution exists and it has the following properties [10] :

$$\pi_i = \prod_{k=1}^{i} \frac{p_{k-1}}{q_k}\pi_0 \text{ and } \sum_{i=0}^{N} \pi_i = 1.$$

The expected number of agents that adopted the advertised item is the expected value of this distribution π.

$$\mathbb{E}[\pi] = \sum_{i=0}^{N} i\pi_i = \pi_0 \sum_{i=0}^{N} i \prod_{k=1}^{i} \frac{p_{k-1}}{q_k}.$$

This mathematical expression can be calculated by setting the values of each p_{k-1} and q_k. $\mathbb{E}[\pi]$ gives us the expected number of infected agents in the population. Hence, the expected fame of the advert is :

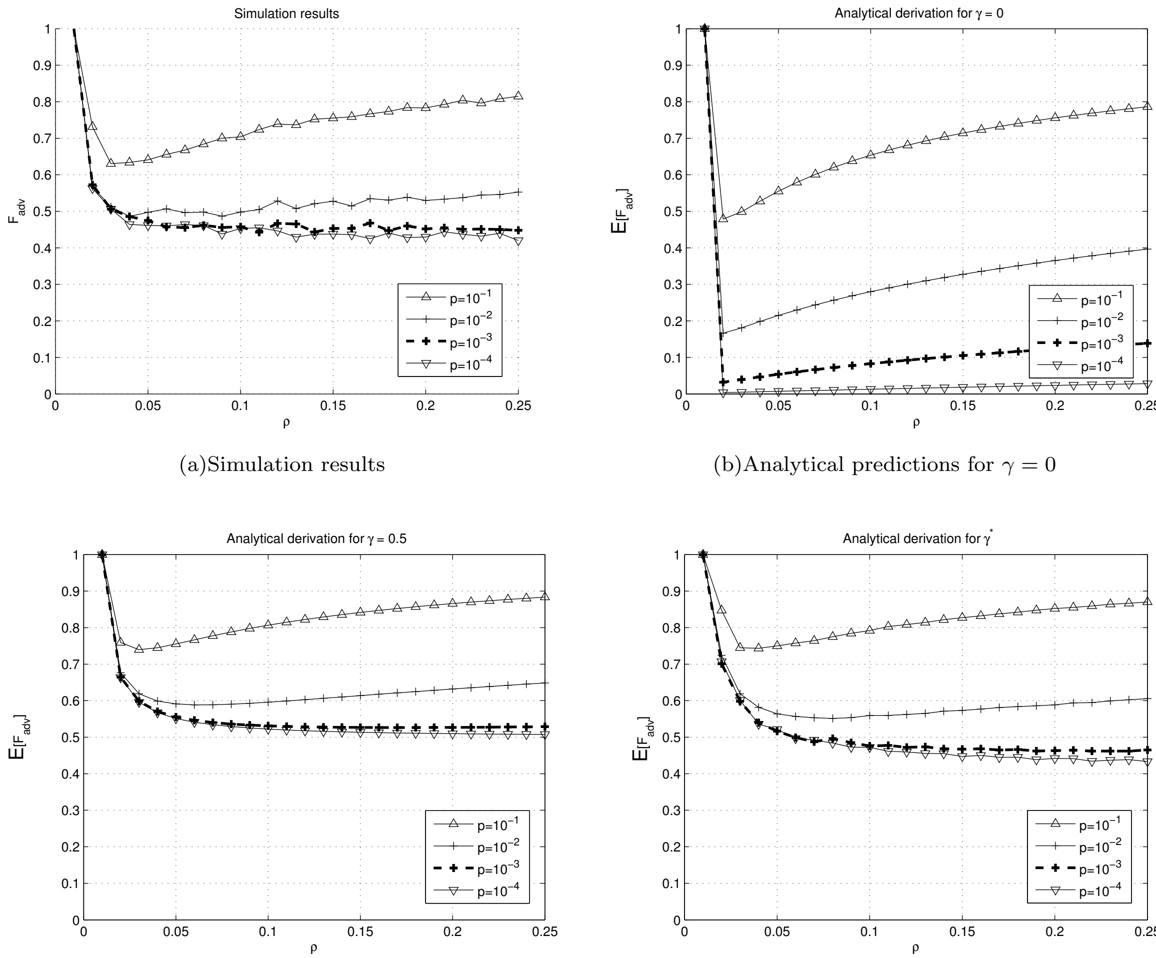
$$\mathbb{E}[F_{adv}] = \frac{\mathbb{E}[\pi]}{N}$$

$\mathbb{E}[\pi]$ clearly depends on the parameters N, M, p and γ. Except γ, the other parameters are known. So we should focus on the value of γ.

D. Discussion about γ

γ was the probability that the taker already knows the item recommended by the giver. It was tractable for $M=1$ as it was discussed in Sec. IV B.

The most crucial question is how to determine the exact value of γ for $M > 1$. Since memories of each agent progress on the basis of their own history and γ depends



(a)Simulation results

(b)Analytical predictions for $\gamma = 0$

(c)Analytical predictions for $\gamma = 0.5$

(d)Analytical predictions for γ^*

FIG. 5: Simulation results vs. the analytical predictions on the fame of the advertised item, for reasonable memory sizes of M from 1 to 25, are depicted. Theoretical results rendered by the Markov chain model closely reflect the simulation except for $\gamma = 0$. Simulation results are best reflected via γ^*.

on the memory content of specific g and t, the exact value of γ is almost intractable for $M > 1$. Hence, we choose to set the value of γ in an empirical way.

Since each agent interacts with each other, it is obvious that what g and t know are going to intersect as time goes by. Hence, setting γ to zero, is not realistic. Nonetheless, we present analytical derivations for $\gamma = 0$, too in Fig. 5(b). Since determining the exact value of γ seems to be a challenging issue, for the present, one can simply set γ to 0.5. Results can be seen in Fig. 5(c).

On the other hand, it seems logical to approximate γ with the fraction of items that taker knows over the total number of items, $I = N$. If so, the recommended item would be already known by taker with probability $\frac{M}{N}$. Nevertheless, in this consideration we ignore the completely forgotten items. A better approximation would be made if we consider γ as $\frac{M}{U}$ where U represents the number of unforgotten items. To do this, we make use of the logs generated by the 100 realizations of our previous simulation and take an average. Formally, we construct the estimator as follows:

$$\gamma^* = \frac{1}{R}\sum_{i=1}^{R}\frac{M}{U_i}$$

where U_i represents the number of unforgotten agents at the end of the ith run and R represents the number of realizations which is equal to 100. Results can be seen in Fig. 5(d). Effect of the forgotten agents would be more influential primarily for small memory sizes. For larger memory sizes, the number of forgotten items decreases, and eventually we have $U = N$.

Note that for $M > 1$, there are many absorbing states. From another point of view, γ also reveals the degree of closeness to an absorbing state. It is evident that at the beginning, $\gamma \geq 0$ and at an absorbing state, $\gamma = 1$.

Consider the case where every agent knows the same M items but not the advert. This cannot be an absorbing state. Due to our settings, there exists always the possibility of adopting the advertisement with probability p. As a result of this, at an absorbing state, the number of agents knowing the advertised item is obliged to be N.

We have set the average number of iteration to a big number such as $\nu = 10^6$, just like in [2]. Even if this provides a considerable amount of interactions among agents, yet it does not suffice all the time to attain an absorbing state, particularly for larger memory size. As a final remark, setting γ to one implies that $q_i = 0$ for all the states.

We can illustrate the expected value of the steady distribution π as a function of the memory size for a fixed p and N. It is not expected for an agent to be informed of all others. So, we show our findings just for reasonable memory size when ρ is in the range of $0.01 - 0.25$. Comparison between the simulation results and the analytical predictions for various $\gamma \in \{0, 0.5, \gamma^*\}$ values can be seen in Fig. 5. Model predictions can quantitatively reproduce the simulation results except for $\gamma = 0$. Especially, our experimental results shown in Fig. 5(a) and analytical derivation for γ^* shown in Fig. 5(d) coincide. This visual inspection is also verified by the quadratic loss function. We observe that the system always favors the popularity of the advertisement for greater p. On the other hand, for lesser p, the system can still significantly favor the popularity of the advertisement, particularly in the case of small memory size. We observe that in the extreme case when memory size is equal to 1, regardless of p, every agent in the population adopts the intimation of the advertisement which can be interpreted as a great success.

V. CONCLUSIONS

Word-of-mouth marketing methods are investigated in its simplest form by [2]. Our model extends previous work in two important ways. In addition to word-of-mouth spreading, we also incorporate an advertisement mechanism. Moreover, we also provide an analytical explanation that can easily predict the steady state of the system by specifying a corresponding Markov Chain. The analytical results agree with the simulations. This model constructs a theoretical framework for studying the propagation of any phenomena. In this work, we primarily focus on how popularity spread among the individuals.

It is important to determine appropriate conditions for a company before make investments about its products. That is why, detecting the behavior of our system is important. We considered that there exists only one advertised item, namely $A = 1$. In the general case, we found that the system favors the advertised item for greater p. Additionally, in the extreme case, where the agents memory size is close to one, a single advertised item can have a significant effect on the whole population. Thus, the number of infected agents performs a random walk, until the system reaches its absorbing state, S_N where all agents adopt the advertised item.

As a future work, we plan to generalize our problem to the case where $A \geq 1$ and to any network topology. It would be interesting to study the simultaneous presence of more than one advertised item which can be interpreted as a competition among firms.

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